6. LIN'KOV R.V., Slow motions of a conducting gyroscope in resonant interaction with an alternating magnetic field, Zh. tekhn. fiziki, 50, 6, 1980.

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BRANCHING AND STABILITY OF PERMANENT ROTATIONS AND RELATIVE EQUILIBRIA OF A BODY SUSPENDED FROM A ROD*

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The problem of the motion of a rigid body with a triaxial central ellipsoid of inertia suspended from a fixed point of a weightless non-deformable rod whose point of contact with the body lies on the principal central axis of inertia, is considered. Sets of all permanent rotations and relative equilibria of the body, their branching and stability, are studied. The results are presented in the form of bifurcation diagrams. The distribution of permanent rotations (relative equilibria) on these diagrams obeys the law of variation of stability when the value of the area integration constant (the angular velocity of the translational rotation of the body) is fixed.

The permanent rotations and relative equilibria of a body suspended on a string were studied in /1-3/.

1. Let us consider the motion of a body suspended on a hinge from a weightless non-deformable rod attached to a fixed point O_1 with the point of suspension O lying on the principal central axis of inertia.

The equations of motion of the body admit of energy and area integrals, and we have the following expression /4/ for the changed potential energy of the system:

$$W = {}^1/_2\ k^2J^{-1} + \Pi$$
$$\Pi = -mg(l\mathbf{v} - \mathbf{a})\cdot\boldsymbol{\varkappa}, \quad J = \boldsymbol{\varkappa}\cdot\boldsymbol{\Theta}\cdot\boldsymbol{\varkappa} + m[\boldsymbol{\varkappa}\times(l\mathbf{v} - \mathbf{a})]^2$$

Here k is the constant of the area integral, Π is the potential energy due to gravity, J is the moment of inertia of the body about the vertical passing through the point O_1, m and $\boldsymbol{\Theta}$ is the mass and central tensor of inertia of the body with diagonal elements J_1, J_2, J_3, $\boldsymbol{\varkappa}$ and $\mathbf{v}$ are unit vectors of the descending vertical and the direction of the string from the point O_1 to O, $\mathbf{a}$ is the radius vector of the point O relative to the centre of mass C of the body, and g and l are the acceleration due to gravity and the length of the rod.

We introduce two right rectangular coordinate systems: the system $Cx_1x_2x_3$ rigidly attached to the body, whose axes coincide with the principal central axes of inertia, and the system $O_1y_1y_2y_3$ rotating with angular velocity $\Omega = kJ^{-1}$ about the y_3 axis directed vertically downwards.

We shall assume that the point O at which the rod is joined to the body, lies on the x_3 axis whose direction coincides with the direction of the vector $\mathbf{a}$. We shall denote by ν_s the projections of the vector $\mathbf{v}$ on to the y_s $(s = 1, 2, 3)$ axes. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the unit vectors of the x_1, x_2, x_3 axes and $\alpha_s, \beta_s, \gamma_s$ their projections on to the y_s axes, and

$$\pi_\alpha = \boldsymbol{\alpha}^2 - 1 = 0, \quad \pi_\beta = \boldsymbol{\beta}^2 - 1 = 0, \quad \pi_\gamma = \boldsymbol{\gamma}^2 - 1 = 0, \quad \pi_\nu = \mathbf{v}^2 - 1 = 0 \tag{1.1}$$
$$\pi_{\alpha\beta} = \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0, \quad \pi_{\beta\gamma} = \boldsymbol{\beta}\cdot\boldsymbol{\gamma} = 0, \quad \pi_{\gamma\alpha} = \boldsymbol{\gamma}\cdot\boldsymbol{\alpha} = 0$$

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We have the following expressions for Π and J in the $O_1y_1y_2y_3$ coordinate system:

$$\Pi = -mg\,(lv_3 - a\gamma_3),\quad J = J_1\alpha_3^2 + J_2\beta_3^2 + J_3\gamma_3^2 + m\ \ [(lv_1 - a\gamma_1)^2 + (lv_2 - a\gamma_2)^2]$$

Further, instead of W we shall consider the function

$$W_* = W + {}^1/{}_2\Omega^2(\lambda_\nu\pi_\nu + \lambda_\alpha\pi_\alpha + \lambda_\beta\pi_\beta + \lambda_\gamma\pi_\gamma + 2\lambda_{\alpha\beta}\pi_{\alpha\beta} + 2\lambda_{\beta\gamma}\pi_{\beta\gamma} + 2\lambda_{\gamma\alpha}\pi_{\gamma\alpha}),\quad \Omega = kJ^{-1}$$

where $\lambda_\nu, \lambda_\alpha, \ldots, \lambda_{\gamma\alpha}$ are the undetermined Lagrange multipliers.

2. The condition of stationarity

$$\delta W_* = 0 \tag{2.1}$$

of the function W_* with respect to the variables introduced, together with (1.1), leads to the equations for determining the stationary motions of the body, representing its uniform rotations about the vertical with angular velocity $\Omega_0 = k_0J_0^{-1}$, where k_0, J_0 are the values of k, J for the stationary motion /4/.

From (2.1) we obtain (the relations not written out are obtained by circular permutation of the symbols within the brackets)

$$\begin{aligned}
&\lambda_\alpha = J_1\alpha_3^2,\ \lambda_{\alpha\beta} = J_1\alpha_3\beta_3,\ \lambda_{\alpha\gamma} = J_1\alpha_3\gamma_3\ (\alpha\beta,\ 12)\\
&\lambda_{\gamma\alpha} = J_3\gamma_3\alpha_3 - mga\alpha_3\Omega^{-2} - ma\ \ [l\,(\nu_1\alpha_1 + \nu_2\alpha_2) - a\,(\gamma_1\alpha_1 + \\
&\quad\gamma_2\alpha_2)]\ \ (\alpha\beta)\\
&\lambda_\gamma = J_3\gamma_3^2 - mg\gamma_3\Omega^{-2} - ma\ [l\,(\nu_1\gamma_1 + \nu_2\gamma_2) - a\,(\gamma_1^2 + \gamma_2^2)]\\
&\nu_1 = -\frac{mal}{\lambda_\nu^*}\gamma_1(12),\quad \nu_3 = \frac{mgl}{\lambda_\nu\Omega^2},\quad \lambda_\nu^* = \lambda_\nu - ml^2
\end{aligned}\tag{2.2}$$

Relations (2.2) for ν_1, ν_2 imply that in the case of stationary motions the rod and x_3 axis lie in the same vertical plane.

Taking into account (2.2), we obtain from the relation $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$ the relation $(J_1 - J_2)\alpha_3\beta_3 = 0$, which leads, when $J_1 \neq J_2$, to the following two cases: $\alpha_3 = 0$ and $\beta_3 = 0$.

Let us consider the case $\alpha_3 = 0$. Taking into account (2.2) we obtain, from $\lambda_{\beta\gamma} = \lambda_{\gamma\beta}$, the relation which leads to the analysis of two subcases

$$\begin{aligned}
&\beta_3 = 0\ \text{ or }\ \gamma_3 = -\frac{mga\lambda_\nu^*}{\Omega^2[(\Delta_{23} + ma^2)\lambda_\nu^* + (mal)^2]}\\
&(\Delta_{j3} = J_j - J_3,\quad j = 1, 2)
\end{aligned}\tag{2.3}$$

The assumption that $\beta_3 = 0$, yields the solutions

$$\begin{aligned}
&\alpha_1 = \beta_2 = 1,\ \ \gamma_3 = \pm 1,\ \ \nu_3 = \pm 1,\ \ \alpha_2 = \alpha_3 = \beta_3 = \beta_1 = \\
&\quad\gamma_1 = \gamma_2 = \nu_1 = \nu_2 = 0\\
&\lambda_\alpha = \lambda_{\alpha\beta} = \lambda_{\alpha\gamma} \equiv \lambda_\beta = \lambda_{\beta\gamma} = 0,\quad \lambda_\nu^* = (mal)^2\sigma^{-1}\\
&\sigma = ma^2\Omega^2 l\,(\nu_3 g - \Omega^2 l)^{-1},\ \ \lambda_\gamma = J_3 - \gamma_3\,mga\Omega^{-2},\ \ k = J_3\Omega
\end{aligned}\tag{2.4}$$

The solutions (2.4) describe four one-parameter families of uniform rotations of the body, in which the rod and the x_3 axis are both vertical, the point O can lie below $(\nu_3 = 1)$, as well as above $(\nu_3 = -1)$ the point O_1, and the centre of mass C below $(\gamma_3 = -1)$, or above $(\gamma_3 = 1)$ the point O. Fig.1 shows the relative positions of the points O_1, O, C in the uniform rotations (2.4).

The second assumption of (2.3) leads to the solution

$$\begin{aligned}
&\nu_2 = \gamma_2 = \beta_2 = \alpha_1 = \alpha_3 = 0,\ \ \alpha_2 = 1,\ \ \beta_1 = \gamma_3,\ \ \beta_3 = -\gamma_1,\\
&\gamma_1^2 = 1 - \gamma_3^2,\\
&\gamma_3 = -\frac{mga}{\Omega^2(\sigma_* + \Delta_{23})},\quad \nu_1 = -\frac{\sigma}{mal}\gamma_1,\quad \nu_3 = \frac{g\sigma}{\Omega^2 l\sigma_*},\\
&\sigma_* = \sigma + ma^2\\
&\lambda_\alpha = \lambda_{\alpha\beta} = \lambda_{\alpha\gamma} = 0,\quad \lambda_\beta = J_2\beta_3^2,\quad \lambda_{\beta\gamma} = J_2\beta_3\gamma_3,\quad \lambda_\nu^* = ma^2\sigma^{-1}\\
&\lambda_\gamma = \sigma_*\left\{1 + \frac{J_2\sigma_*[(mal)^2 - \sigma^2]}{\Delta_{23}(2\sigma_* + \Delta_{23})\sigma^2}\right\},\quad \Omega^4 = \frac{\Delta_{23}(mga\sigma)^2(2\sigma_* + \Delta_{23})}{\sigma_*^2(\sigma_* + \Delta_{23})^2[(mal)^2 - \sigma^2]}\\
&J = J_3 + \left(\Delta_{23} + \frac{\sigma_*^2}{ma^2}\right)\beta_3^2,\ \beta_3^2 = \frac{\sigma^2(\sigma_* + \Delta_{23})^2 - (mal\sigma_*)^2}{\Delta_{23}(2\sigma_* + \Delta_{23})\sigma^2},\ \ k = J\Omega
\end{aligned}\tag{2.5}$$

Let us now consider the case $\beta_3 = 0$, when the relation $\lambda_{\alpha\gamma} = \lambda_{\gamma\alpha}$, taking both (2.2) and (1.1) into account, yields a relation which leads to the following two subcases:

$$\alpha_3 = 0\ \text{ or }\ \gamma_3 = -\frac{mga\lambda_\nu^*}{\Omega^2[(\Delta_{13} + ma^2)\lambda_\nu^* + (mal)^2]}$$

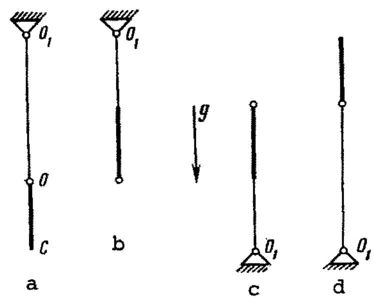

Fig.1

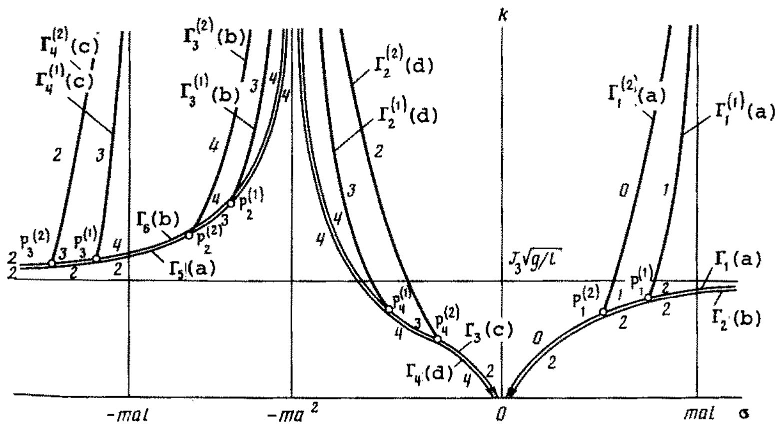

Fig.2

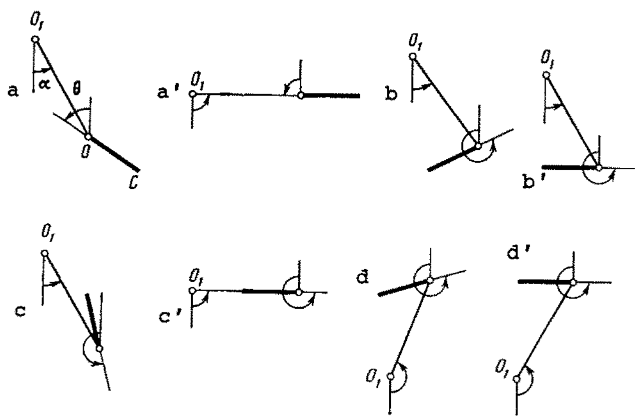

Fig.3

The first case leads to solutions (2.4), and the second case leads to a solution which will be described by the formulas (2.5) in which the symbols ($\alpha\beta$, 12) have been circularly permuted. We shall call this solution (2.6).

Solutions (2.5) and (2.6) describe the one-parameter families of uniform rotations of the system, as a single rigid body, about the vertical, with angular velocity Ω, in which the rod and the x_3 axis both lie in the $O_1y_1y_3$ plane. The x_1 axis for (2.5) and x_2 axis for (2.6) are perpendicular to this plane.

3. The stationary motions (2.4)-(2.6) can be represented geometrically, in k, σ, λ_γ parameter space, by the points on the curve Γ, whose branches, corresponding to the motions

(2.4), are described in parametric form by equations of the form $k = J_3\Omega$, $\sigma = \sigma(\Omega)$, $\lambda_\gamma = \lambda_\gamma(\Omega)$, and the branches corresponding to the motions (2.5) and (2.6), by equations of the form $k = J(\sigma)\,\Omega(\sigma)$, $\lambda_\gamma = \lambda_\gamma(\sigma)$. Fig.2 shows the form of the projection of the curve Γ onto the plane $\lambda_\gamma = 0$ for the case when the parameters of the system satisfy the conditions

$$0 < J_2 - J_3 < J_1 - J_3 < ma\,(l-a) \tag{3.1}$$

The conditions hold, in particular, for a homogeneous body stretched in the direction of the x_3 axis and suspended from a fairly long rod.

In Fig.2 the branches Γ_1(a), Γ_2(b), Γ_3(c), Γ_4(d), Γ_5(a), Γ_6(b), for which the letters a, b, c and d denote the types of motions, with the relative positions of the points O_1, O, C depicted in Fig.1, correspond to the motions (2.4). The branches $\Gamma_1^{(j)}$(a), $\Gamma_2^{(j)}$(d), $\Gamma_3^{(j)}$(b), $\Gamma_4^{(j)}$(c), correspond to the motions (2.5) and (2.6), and for (2.5) $j=1$, while for (2.6) $j=2$. Here the letters a, b, c and d refer to the types of relative positions of the rod O_1O and segment CO of the x_3 axis, shown in Fig.3. Figs.3a-d refer to the case when Ω is finite, and Figs. a'-d' to the limiting case when $\Omega = \infty$. We have the following relations in Fig.3 for the cases a-d and a'-d':

$$\text{a) } 0<\alpha<\vartheta<\frac{\pi}{2};\ \text{b) } 0<\alpha<2\pi-\vartheta<\frac{\pi}{2};\ \text{c) } 0<\vartheta-\pi<\alpha<\frac{\pi}{2}$$

$$\text{a}')\ \alpha=\vartheta-\frac{\pi}{2};\ \text{b}')\ \alpha=\arcsin\frac{a}{l},\ \ \vartheta=\frac{3\pi}{2};\ \text{c}')\ \alpha=\frac{\pi}{2},\ \ \vartheta=\frac{3\pi}{2}$$

$$\text{d) } 0<\pi-\alpha<2\pi-\vartheta<\frac{\pi}{2};\ \text{d}')\ \alpha=\pi-\arcsin\frac{a}{l},\ \ \vartheta=\frac{3\pi}{2}$$

Note that the projections in Fig.2 of the branches of Γ, corresponding to the motions (2.4) on to the plane $\lambda_\gamma = 0$, should be regarded as double curves consisting of two "edges", with a different type of motion shown in Fig.1 corresponding to each edge.

4. The sufficient conditions for stability of the stationary motions (2.4)-(2.6) relative to the variables $\mathbf{v}, \boldsymbol{\omega}, \nu, \alpha, \beta, \gamma$, where $\mathbf{v}$ and $\boldsymbol{\omega}$ are the velocity vectors of the centre of mass and the instantaneous angular velocity of the body, are obtained from Routh's theorem /5/ as the conditions of positive definiteness of the second variation $\delta^2 W_*$ on a linear manifold defined by the equations

$$\delta\pi_\nu = \delta\pi_\alpha = \delta\pi_\beta = \delta\pi_\gamma = \delta\pi_{\alpha\beta} = \delta\pi_{\beta\gamma} = \delta\pi_{\gamma\alpha} = 0 \tag{4.1}$$

Let us denote by $(\delta^2 W_*)$ the value of $\delta^2 W_*$ on the manifold (4.1). When the motion is perturbed, we retain the former values of the variables which vanished when the motion was unperturbed. In this case we will have the following relation for the motion (2.4):

$$\Omega^{-2}(\delta^2 W_*) = \sum_{j=1}^{2}[\lambda_\nu^* \nu_j^2 + 2mal\nu_j\gamma_j + (\lambda_\gamma - J_j - ma^2)\gamma_j^2]$$

The conditions of positive definiteness of this quadratic form are given by the inequalities

$$\lambda_\nu^* > 0,\ \lambda_\nu^*(\lambda_\gamma - J_j - ma^2) - (mal)^2 > 0\ \ (j=1,2)$$

or, after substituting the values for λ_ν^* and λ_γ given by (2.4),

$$\sigma > 0,\ \ \frac{1}{\sigma}\left(\Delta_{j3} + ma^2 + \frac{\gamma_3 mga}{\Omega^2}\right) + 1 < 0\ \ \ (j=1,2)$$

Substituting into these inequalities the value of σ given by (2.4), we can write them thus

$$\begin{aligned}&\sigma = ma^2\Omega^2 l\,(\nu_3 g - \Omega^2 l)^{-1} > 0,\ \ Q_j(\nu_3,\ \gamma_3,\ \Omega^2) > 0\ \ \ (j=1,\ 2)\\ &Q_j = \Delta_{j3} l\Omega^4 - \nu_3 g\,(\Delta_{j3} + ma^2 - \nu_3\gamma_3\,mal)\,\Omega^2 -\\ &\quad \nu_3\gamma_3 g^2 ma\ \ (\nu_3 = \pm 1,\ \gamma_3 = \pm 1)\end{aligned} \tag{4.2}$$

When analysing conditions (4.2), we shall assume, to be specific, that the inequalities (3.1) hold for the parameters of the system.

Let us denote by $P_i^{(1)}$ and $P_i^{(2)}$ $(i = 1, 2, 3, 4)$ the points on the curve Γ (Fig.2) at which the motions (2.5) and (2.6) respectively branch out from the motions (2.4). The values σ_{ji}, Ω_{ji}^2 $(j=1,2)$ of the parameters σ, Ω^2 are given for these points by the equations

$$\nu_3 = \frac{g\sigma}{\Omega^2 l\sigma_*},\ \ \ \gamma_3 = -\frac{mga}{\Omega^2(\sigma_* + \Delta_{j3})}\ \ (\nu_3 = \pm 1,\ \gamma_3 = \pm 1,\ j = 1, 2)$$

or, substituting into the second equation the value of σ from the first equation, we obtain

$$\sigma = \frac{\nu_3 m a^2 \Omega^2 l}{g - \nu_3 \Omega^2 l}, \quad Q_j(\nu_3, \gamma_3, \Omega^2) = 0 \ (\nu_3 = \pm 1, \ \gamma_3 = \pm 1, \ j = 1, 2$$

Let us consider the motions (2.4) in which the point O lies below the point O_1 $(\nu_3 = 1)$, and the point C below the point O $(\gamma_3 = -1)$ (Fig.1a). We will denote by $0 < \Omega_{21}^2 < \Omega_{11}^2 < g/l < \Omega_{22}^2 < \Omega_{12}^2$ the positive roots of the equations $Q_j(1, -1, \Omega^2) = 0$ $(j = 1, 2)$. Then the analysis of condition (4.2) will lead to the following conclusions. The motions in question are stable and the degree of instability $\chi = 0$ if $0 < \Omega^2 < \Omega_{21}^2$, and are unstable and $\chi = 1$ if $\Omega_{21}^2 < \Omega^2 < \Omega_{11}^2$, and $\chi = 3$ if $\Omega_{22}^2 < \Omega^2 < \Omega_{12}^2$. We have here $\chi = 2$ if $\Omega_{11}^2 < \Omega^2 < \Omega_{22}^2$, and $\chi = 4$ if $\Omega_{12}^2 < \Omega^2 < \infty$.

Let us now consider the motions (2.4) in which point O lies below the point O_1 $(\nu_3 = 1)$, and point C lies above the point O $(\gamma_3 = 1)$. We will denote by $g/l < \Omega_{23}^2 < \Omega_{13}^2$ the positive roots of the equations $Q_j(1, 1, \Omega^2) = 0$ $(j = 1, 2)$. Then for these motions we will have $\chi = 2$ if $0 < \Omega^2 < \Omega_{23}^2$, $\chi = 3$, if $\Omega_{23}^2 < \Omega^2 < \Omega_{13}^2$, and $\chi = 4$ if $\Omega_{13}^2 < \Omega^2 < \infty$.

Next we consider the motions (2.4) in which point O lies above the point O_1 $(\nu_3 = -1)$, and point C below the point O $(\gamma_3 = -1)$ (Fig.1c). We denote by $0 < \Omega_{24}^2 < \Omega_{14}^2$ the positive roots of the equations $Q_j(-1, -1, \Omega^2) = 0$ $(j = 1, 2)$. Then we shall have for these motions $\chi = 2$ if $0 < \Omega^2 < \Omega_{24}^2$, $\chi = 3$ if $\Omega_{24}^2 < \Omega^2 < \Omega_{14}^2$ and $\chi = 4$ if $\Omega_{14}^2 < \Omega^2 < \infty$.

Finally we consider the motion (2.4) in which point O lies above the point O_1 $(\nu_3 = -1)$ and point C lies above the point O $(\gamma_3 = 1)$ (Fig.1d). Then for these motions $\chi = 4$ for all $\Omega^2 > 0$.

The stationary motions for which $\chi = 0$ are stable and the motions for which $\chi = 1$ or $\chi = 3$ are unstable. The character of the stability of the motions for which $\chi = 2$ or $\chi = 4$, cannot be determined using Routh's theorem.

5. When studying the stability of the motions (2.5) and (2.6), we shall assume without loss of generality that during the motion of the system the rod remains, at all times, within the plane $y_2 = 0$ and therefore $\nu_2 \equiv 0$.

Let us introduce the function

$$\begin{aligned}
&W^*(\mu, \lambda, \varkappa, \gamma_1, \nu_1, \gamma_3, \nu_3, \alpha_3, \beta_3, \gamma_2) = W + {}^1/_2\Omega^2(\mu V_1 + \\
&\quad \lambda V_2 + \varkappa V_3); \\
&V_1 = \nu_1^2 + \nu_3^2 - 1 = 0, \ V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0, \\
&V_3 = \alpha_3^2 + \beta_3^2 + \gamma_3^2 - 1 = 0, \ \Omega = kJ^{-1}
\end{aligned}$$

where $\mu, \lambda, \varkappa$ are the undetermined Lagrange multipliers. The values of the variables $q = (\mu, \lambda, \varkappa, \gamma_1, \nu_1, \gamma_3, \nu_3, \alpha_3, \beta_3, \gamma_2)$, obtained from the condition $\partial W^*/\partial q = 0$ of stationarity of the function W^* are identical with their values in the solutions (2.5) and (2.6), and we have $\mu = \lambda_\nu$, $\lambda = \sigma_*$, while $\varkappa = J_2$ for (2.5) and $\varkappa = J_1$ for (2.6).

The sufficient conditions of stability of the motions (2.5) and (2.6) are obtained as the conditions of positive definiteness of the second variation $\delta^2 W^*$ on the linear manifold $\delta V_1 = \delta V_2 = \delta V_3 = 0$. The conditions reduce to the requirement that the last four principal diagonal minors of the Hessian $D(W^*) = \partial^2 W^*/\partial q^2$ of the function W^* be negative over all its variables /5/.

The sufficient conditions of stability for (2.5) are given by the inequalities /6/

$$\begin{aligned}
&\Omega^2(J_2 - J_1) > 0, \quad \Omega^2\sigma > 0 \qquad (5.1)\\
&\Delta_1' = ml^2\Omega^2\gamma_1^2\gamma_3^2\,[ma^2\sigma^{-1} + \nu_1^2 + 4J^{-1}m\,(l\nu_1 - a\gamma_1)^2\nu_3^2] > 0 \\
&\Delta_1 = \frac{\nu_1^2\beta_3^2 m l^2 \Omega^7 \sigma_*^3 (\sigma_* + \Delta_{23})^3 [(mal)^2 - \sigma^2]}{J\Delta_{23}(mga)^2\sigma^4(2\sigma_* + \Delta_{23})}\frac{dk}{d\sigma} > 0 \\
&D(W^*) = \Omega^4\sigma(J_1 - J_2)\Delta_1
\end{aligned}$$

and for (2.6) by the analogous inequalities in which $J_2 - J_1$ has been replaced by $J_1 - J_2$, β_3 by α_3 and J_2 by J_1.

Let us consider the motions (2.5). For the branch $\Gamma_1^{(1)}$ (Fig.2) we have $\sigma > 0$, $dk/d\sigma > 0$ and conditions (5.1) hold except the first one. Therefore, the motions corresponding to the branch $\Gamma_1^{(1)}$ are unstable $(\chi = 1)$. For $\Gamma_2^{(1)}$ we have $\sigma < 0$, $\sigma + ma^2 > 0$, $dk/d\sigma < 0$, hence $\Delta_1 < 0$. Therefore the motions corresponding to the branch $\Gamma_2^{(1)}$ are unstable $(\chi = 3)$. For $\Gamma_3^{(1)}$ we have $\sigma + ma^2 < 0$, $2(\sigma + ma^2) + J_2 - J_3 > 0$, $dk/d\sigma > 0$ and $\Delta_1 < 0$, therefore the motions corresponding to the branch $\Gamma_3^{(1)}$ are unstable $(\chi = 3)$. Finally, for $\Gamma_4^{(1)}$ we have $2(\sigma + ma^2) + J_2 - J_3 < 0$, $dk/d\sigma > 0$ and $\Delta_1 < 0$, therefore the motions are unstable $(\chi = 3)$.

The conditions of stability of the motions (2.6) can be investigated in exactly the same manner.

Fig.2 shows the results of analysing the conditions of stability of the motions (2.4)-

(2.6). The numbers accompanying the branches of curve Γ indicate the degree of instability of the corresponding motions. The distribution of the stable and unstable motions on the branches of the curve Γ is governed by the law of variation in stability for fixed values of the parameter k, and the change in the degree of instability occurs only at the bifurcation points.

6. The problem of the relative equilibrium of a body suspended from the fixed point O_1 on a rod attached at the point O lying on the principal central axis of inertia, in a $O_1y_1y_2y_3$ coordinate system rotating uniformly about the vertical with angular velocity Ω, is reduced to studying the potential energy $U = \Pi - {}^1/_2 J\Omega^2$ due to gravity and the centrifugal force /4/.

Let us denote by U_* the function defined by the expression for W_* in which W has been replaced by U. The conditions of stationarity of W_* and U_* are the same, and formulas (2.4)-(2.6), in which relations of the form $k = J\Omega$ have been omitted, will describe the families of all relative equilibria.

The set of relative equilibria can be represented geometrically in Ω, σ, λ_γ parameter space by the point on the curve B determined by equations of the form $\Omega = \Omega(\sigma)$, $\lambda_\gamma = \lambda_\gamma(\sigma)$.

When conditions (3.1) hold, the projection of the curve B on to the plane $\lambda_\gamma = 0$ consists of several branches. The branches B_1(a) and B_2(b), B_3(c) and B_4(d), B_5(a) and B_6(b) forming double lines, correspond to the equilibria (2.4). Here the letters a, b, c and d denote the types of equilibria in which the relative positions of the points O_1, O, C, are given in Fig.1. The pairs of branches $B_1^{(1)}$(a) and $B_1^{(2)}$(a), $B_2^{(1)}$(d) and $B_2^{(2)}$(d), $B_3^{(1)}$(b) and $B_3^{(2)}$(b), $B_4^{(1)}$(b) and $B_4^{(2)}$(b) which branch out of the branches B_1(a), B_3(c), B_5(a) and B_6(b) at the points $P_1^{(j)}$, $P_4^{(j)}$, $P_2^{(j)}$, $P_3^{(j)}$ respectively, correspond to the equilibria (2.5) and (2.6). Here the letters a, b, c and d denote the types of relative position of the rod O_1O and the segment CO of the x_3 axis shown in Fig.3.

In order to study the stability of the relative equilibria (2.5) and (2.6), we shall introduce the function U^* obtained from the expression for W^* by replacing W and U. The values of the variables $q = (\mu, \lambda, \varkappa, \gamma_1, \nu_1, \gamma_3, \nu_3, \alpha_3, \beta_3, \gamma_2)$ obtained from the condition $\partial U^*/\partial q = 0$ of stationarity of the function U^* are the same as those for the families of equilibria (2.5), (2.6), and we have $\mu = \lambda_\nu$, $\lambda = \sigma_*$, while $\varkappa = J_2$ for (2.5) and $\varkappa = J_1$ for (2.6).

The sufficient conditions for stability of the relative equilibria (2.5), (2.6) are reduced to the demand that the last four principal diagonal minors of the Hessian $D(U^*) = \partial^2 U^*/\partial q^2$ of the function U^* be negative. In the case of (2.5) the conditions are expressed by the inequalities /6/

$$\Omega^2 (J_2 - J_1) > 0,\ \Omega^2\sigma > 0,\ ml^2\Omega^{14}\gamma_1{}^2\gamma_3{}^2\ (ma^2\sigma^{-1} + \nu_1{}^2) > 0$$

$$\Delta_1 = \frac{\Delta_{23}\gamma_1{}^4\Omega^8 mg^2\sigma^2 (2\sigma_* + \Delta_{23})}{2\sigma_* (\sigma_* + \Delta_{23})} \frac{d\Omega^4}{d\sigma} > 0 \quad D(U^*) = \Omega^4\sigma (J_1 - J_2) \Delta_1$$

and for (2.6) by the analogous inequalities in which $J_2 - J_1$ has been replaced by $J_1 - J_2$ and Δ_{23} by Δ_{13}.

Analysing these conditions we arrive at the following conclusions. The relative equilibria corresponding to the branch $B_1^{(2)}$, are stable ($\chi = 0$). The equilibria corresponding to the branches $B_1^{(1)}$ and $B_2^{(1)}$, $B_3^{(1)}$, $B_4^{(1)}$ are unstable, and we have for them $\chi = 1$ and $\chi = 3$ respectively. For the branches $B_2^{(2)}$, $B_3^{(2)}$, $B_4^{(2)}$ we have $\chi = 2$.

REFERENCES

1. ISHLINSKII A.YU., STOROZHENKO V.A. and TEMCHENKO M.E., On the motion of an axisymmetric solid suspended on a string. Izv. Akad. Nauk SSSR, MTT, 6, 1979.
2. RUMYANTSEV V.V., On the dynamics of a solid suspended on a string. Izv. Akad. Nauk SSSR, MTT, 4, 1983.
3. NOZADZE G.T., On the stability and bifurcation of the positions of relative equilibrium of a solid suspended on a string. Izv. AN SSSR, MTT, 3, 1984.
4. MOISEYEV N.N. and RUMYANTSEV V.V., Dynamics of a body with cavities filled with fluid. Moscow, Nauka, 1965.
5. RUBANOVSKII V.N. and STEPANOV S.YA., On Routh's theorem and the Chetayev method for constructing the Lyapunov function from the integrals of the equations of motion. PMM 3, 5, 1969.
6. RUBANOVSKII V.N., On the bifurcation and stability of stationary motions. Sofiya, Teoretichna i Prilozhna Mekhanika, 5, 1, 1974.

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